

© International Baccalaureate Organization 2021

All rights reserved. No part of this product may be reproduced in any form or by any electronic or mechanical means, including information storage and retrieval systems, without the prior written permission from the IB. Additionally, the license tied with this product prohibits use of any selected files or extracts from this product. Use by third parties, including but not limited to publishers, private teachers, tutoring or study services, preparatory schools, vendors operating curriculum mapping services or teacher resource digital platforms and app developers, whether fee-covered or not, is prohibited and is a criminal offense.

More information on how to request written permission in the form of a license can be obtained from <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

© Organisation du Baccalauréat International 2021

Tous droits réservés. Aucune partie de ce produit ne peut être reproduite sous quelque forme ni par quelque moyen que ce soit, électronique ou mécanique, y compris des systèmes de stockage et de récupération d'informations, sans l'autorisation écrite préalable de l'IB. De plus, la licence associée à ce produit interdit toute utilisation de tout fichier ou extrait sélectionné dans ce produit. L'utilisation par des tiers, y compris, sans toutefois s'y limiter, des éditeurs, des professeurs particuliers, des services de tutorat ou d'aide aux études, des établissements de préparation à l'enseignement supérieur, des fournisseurs de services de planification des programmes d'études, des gestionnaires de plateformes pédagogiques en ligne, et des développeurs d'applications, moyennant paiement ou non, est interdite et constitue une infraction pénale.

Pour plus d'informations sur la procédure à suivre pour obtenir une autorisation écrite sous la forme d'une licence, rendez-vous à l'adresse <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

© Organización del Bachillerato Internacional, 2021

Todos los derechos reservados. No se podrá reproducir ninguna parte de este producto de ninguna forma ni por ningún medio electrónico o mecánico, incluidos los sistemas de almacenamiento y recuperación de información, sin la previa autorización por escrito del IB. Además, la licencia vinculada a este producto prohíbe el uso de todo archivo o fragmento seleccionado de este producto. El uso por parte de terceros —lo que incluye, a título enunciativo, editoriales, profesores particulares, servicios de apoyo académico o ayuda para el estudio, colegios preparatorios, desarrolladores de aplicaciones y entidades que presten servicios de planificación curricular u ofrezcan recursos para docentes mediante plataformas digitales—, ya sea incluido en tasas o no, está prohibido y constituye un delito.

En este enlace encontrará más información sobre cómo solicitar una autorización por escrito en forma de licencia: <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

Mathematics: analysis and approaches
Higher level
Paper 1

Thursday 6 May 2021 (afternoon)

Candidate session number

| | | | | | | | | | |
|--|--|--|--|--|--|--|--|--|--|
| | | | | | | | | | |
|--|--|--|--|--|--|--|--|--|--|

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.



Please **do not** write on this page.

Answers written on this page
will not be marked.



16EP02

2. [Maximum mark: 7]

Solve the equation $2 \cos^2 x + 5 \sin x = 4$, $0 \leq x \leq 2\pi$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

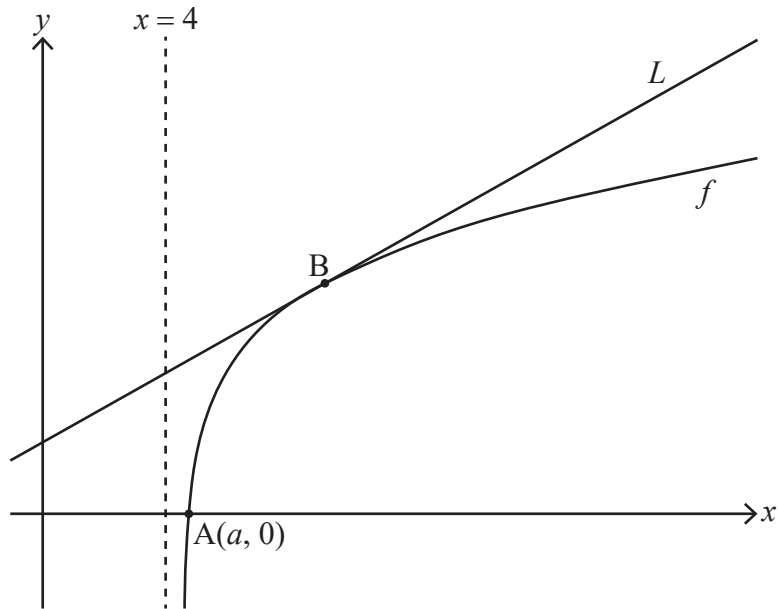
.....



4. [Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x -axis at point A , with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B .



- (a) Find the exact value of a . [3]
- (b) Given that the gradient of L is $\frac{1}{3}$, find the x -coordinate of B . [6]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....



8. [Maximum mark: 8]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \mathbf{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \mathbf{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

- (a) Show that l_1 and l_2 do not intersect. [3]
- (b) Find the minimum distance between l_1 and l_2 . [5]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....



9. [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} dx$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....



16EP11

Turn over

Do **not** write solutions on this page.

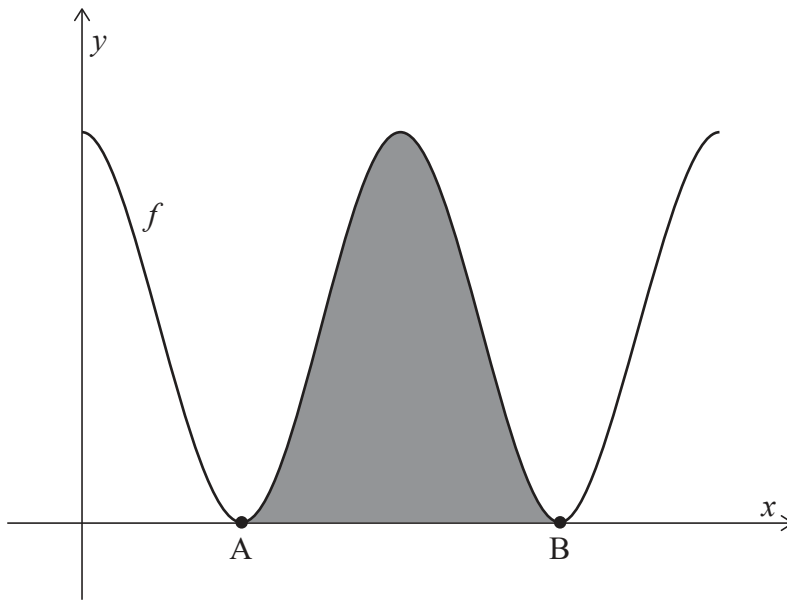
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6 \cos x$, for $0 \leq x \leq 4\pi$.

The following diagram shows the graph of $y = f(x)$.



The graph of f touches the x -axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x -axis, between the points A and B.

- (a) Find the x -coordinates of A and B. [3]
- (b) Show that the area of the shaded region is 12π . [5]

(This question continues on the following page)



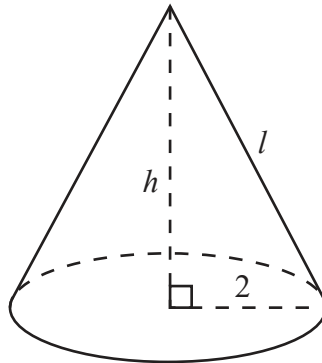
Do **not** write solutions on this page.

(Question 10 continued)

The right cone in the following diagram has a total surface area of 12π , equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h , and slant height l .

diagram not to scale



- (c) Find the value of l . [3]
- (d) Hence, find the volume of the cone. [4]



Do **not** write solutions on this page.

11. [Maximum mark: 20]

The acceleration, $a \text{ ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v \text{ ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0 \text{ ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)e^{-t} - 1$. [6]

(b) Initially at O , the particle moves in the positive direction until it reaches its maximum displacement from O . The particle then returns to O .

Let s metres represent the particle's displacement from O and s_{\max} its maximum displacement from O .

(i) Show that the time T taken for the particle to reach s_{\max} satisfies the equation $e^T = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for s_{\max} in terms of v_0 . [7]

Let $v(T - k)$ represent the particle's velocity k seconds before it reaches s_{\max} , where

$$v(T - k) = (1 + v_0)e^{-(T-k)} - 1.$$

(c) By using the result to part (b) (i), show that $v(T - k) = e^k - 1$. [2]

Similarly, let $v(T + k)$ represent the particle's velocity k seconds after it reaches s_{\max} .

(d) Deduce a similar expression for $v(T + k)$ in terms of k . [2]

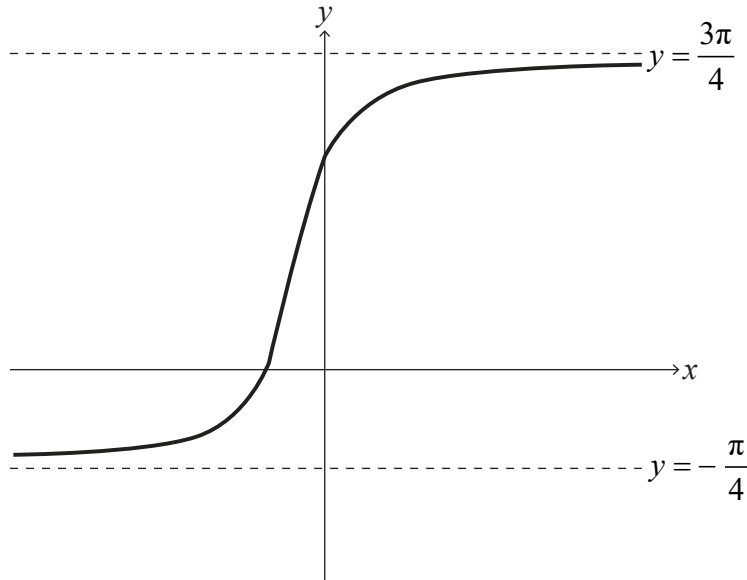
(e) Hence, show that $v(T - k) + v(T + k) \geq 0$. [3]



Do **not** write solutions on this page.

12. [Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



- (a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]
- (b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$. [4]
- (c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}, x > 0$. [3]
- (d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^n \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

References:

© International Baccalaureate Organization 2021



16EP15

Please **do not** write on this page.

Answers written on this page
will not be marked.



16EP16